THE BIFURCATION OF FLUID FLOW BETWEEN ROTATING CYLINDERS

(O BIFURKATSII TECHENII ZHIDKOSTI MEZHDU VRASHCHAIUSHCHIMISIA TSILINDRAMI)

PMM Vol.30, № 4, 1966, pp.768-773

Iu.P.IVANILOV and G.N.IAKOVLEV
(Moscow)

(Received January 10, 1966)

An analysis is presented of the generation of secondary steady state fluid flows between two unidirectionally rotating cylinders.

1. Let a homogeneous fluid be contained in the space between two coaxial rotating cylinders. We introduce a system of cylindrical coordinates (r, θ, x) with its x-axis coinciding with the common axis of the two cylinders. We shall attempt to find such steady state axisymmetric flows, the motion of which is independent of angle θ and time. We select the angular velocity ω_1 and the radius of the inner cylinder r_1 as the characteristic dimensions. With this, the dimensionless angular velocity and the radius of the inner cylinder will be equal to unity. We denote the dimensionless angular velocity and the radius of the outer cylinder by ω and a, respectively. The problem thus stated has the known solution [1]

$$v_r = v_x = 0, \quad v_\theta = u(r) = -\alpha r\left(1 + \frac{\beta}{r^2}\right), \quad \alpha = \frac{1 - a^2\omega}{a^2 - 1}, \quad \beta = \frac{a^2(1-\omega)}{a^2\omega - 1} \qquad (1.1)$$

Here, v_r, v_x, v_θ are velocity components.

The flow conforming to solution (1.1) is called the Couette flow. The existence of steady state axisymmetric flows, differing from that defined by (1.1), was proved experimentally by Taylor [2 and 3]. In order to find such flows we shall introduce the stream function ψ, and express v_θ by Formulas

$$v_r = -\frac{1}{Rr}\frac{\partial}{\partial x}(r^2\psi), \qquad v_x = \frac{1}{Rr}\frac{\partial}{\partial r}(r^2\psi), \qquad v_\theta = v + rv, \qquad R = \frac{\omega_1 r_1^2}{\nu} \qquad (1.2)$$

where R is the Reynolds number. The unknown functions ψ and v are defined by the following system of equations:

$$\Lambda\frac{1}{r^3}\Lambda\psi = \lambda\alpha u r^2\frac{\partial v}{\partial x} + \left\{r^2\frac{\partial(r^{-3}\Lambda\psi, r^2\psi)}{\partial(x, r)} + 2r^3 v\frac{\partial v}{\partial x}\right\}$$

$$\Lambda v = r^3\frac{\partial\psi}{\partial x} + \frac{\partial(r^2 v, r^2\psi)}{\partial(x, r)} \qquad (1.3)$$

$$\lambda = 4R^2, \qquad \Lambda \equiv \frac{\partial}{\partial r}r^3\frac{\partial}{\partial r} + r^3\frac{\partial^2}{\partial x^2}, \qquad \frac{\partial(\psi, \varphi)}{\partial(x, r)} \equiv \frac{\partial\psi}{\partial x}\frac{\partial\varphi}{\partial r} - \frac{\partial\psi}{\partial r}\frac{\partial\varphi}{\partial x}$$

$$v = \psi = \partial\psi/\partial r = 0 \quad \text{for} \quad r = 1,\ a$$

trivial solution of problem (1.3) yields the Couette flow. It is well known that system (1.3) has, for sufficiently small values of R a unique solution, namely $v = \psi = 0$.

Together with (1.3) we shall consider a system of linearlized equations of steady axisymmetric flows

$$\Lambda \frac{1}{r^3}\Lambda\psi = \lambda a u r^2 \frac{\partial v}{\partial x}, \quad \Lambda v = r^3\frac{\partial \psi}{\partial x}, \quad v = \psi = \frac{\partial \psi}{\partial r} = 0 \quad \text{for} \quad r = 1, a \tag{1.4}$$

2. We shall define certain functional domains.

Let $H_0(k)$ be a multiplicity of functions ψ defined in the band $\Omega = (1 \leqslant r \leqslant a, \ -\infty < x < \infty)$, having continuous second derivatives in Ω, odd periodic of period $2\pi k^{-1}$ with respect to x and vanishing in the neighborhood of the boundary $\partial\Omega$ (for $r = 1, a$).

We shall denote by $H(k)$ the Hilbertian space obtained by supplementing $H_0(k)$ in the norm generated by scalar multiplication

$$(\psi, \varphi)_H = \int_{-\pi/k}^{\pi/k}\int_1^a \frac{1}{r^3}\Lambda\psi\Lambda\varphi\, dx\, dr = \iint \Big(\frac{1}{r^3}\frac{\partial}{\partial r} r^3 \frac{\partial\psi}{\partial r}\frac{\partial}{\partial r} r^3 \frac{\partial \varphi}{\partial r} + {}$$
$$+ 2r^3 \frac{\partial^2\psi}{\partial x\,\partial r}\frac{\partial^2\varphi}{\partial x\,\partial r} + r^3 \frac{\partial^2\psi}{\partial x^2}\frac{\partial^2\varphi}{\partial x^2}\Big)\, dx\, dr \tag{2.1}$$

Similarly, let $M_0(k)$ be a multiplicity of functions v defined in Ω having continuous first derivatives in Ω, even, periodic of period $2\pi k^{-1}$ with respect to x, and vanishing in the neighborhood of $\partial\Omega$.

We denote by $M(k)$ the Hilbertian space obtained by supplementing $M_0(k)$ in the norm generated by scalar multiplication

$$(v, \vartheta)_M = \int_{-\pi/k}^{\pi/k}\int_1^a r^3\Big(\frac{\partial v}{\partial r}\frac{\partial\vartheta}{\partial r} + \frac{\partial v}{\partial x}\frac{\partial \vartheta}{\partial x}\Big)\, dx\, dr \tag{2.2}$$

L e m m a 2.1 . Norms defined by (2.1) and (2.2) in spaces $H(k)$, $M(k)$ are equivalent to the usual norms W_2^2 and W_2^1.

L e m m a 2.2 . Norm (2.2) is equivalent to norm

$$\| v \|_{M_1}^2 = \int_{-\pi/k}^{\pi/k}\int_1^a r\Big[\Big(\frac{\partial r^2 v}{\partial r}\Big)^2 + \Big(\frac{\partial r^2 v}{\partial x}\Big)^2\Big]\, dx\, dr$$

We shall indicate the path for providing Lemma 2.1. For function from H_0 we have

$$\iint \frac{1}{r^3}\frac{\partial}{\partial r}\Big(r^3\frac{\partial\psi}{\partial r}\Big)\, dx\, dr = \iint r^3 \Big(\frac{\partial^2\psi}{\partial r^2}\Big)^2 dx\, dr + 3\iint r\Big(\frac{\partial\psi}{\partial r}\Big)^2 dx\, dr \tag{2.3}$$

For odd functions $\psi(r, 0) = 0$. Therefore

$$\psi(r, x) = \int_0^x \frac{\partial\psi(r, t)}{\partial t}\, dt$$

which yields

$$\iint |\psi|^2\, dx\, dr \leqslant \iint\int_0^x \Big|\frac{\partial\psi}{\partial t}\Big|^2 dt\, dx\, dr \leqslant \pi k^{-1} \iint \Big(\frac{\partial\psi}{\partial x}\Big)^2 dx\, dr \tag{2.4}$$

Similarly,

$$\iint\Big(\frac{\partial\psi}{\partial x}\Big)^2 dx\, dr \leqslant C\iint \Big(\frac{\partial^2\psi}{\partial x\,\partial r}\Big)^2 dx\, dr \tag{2.5}$$

Equality (2.3) with inequalities (2.4) and (2.5) make it possible to evaluate $\|\psi\|_H$ by means of $\|\psi\|_{W_2^2}$, and vice versa.

The proof of Lemma 2.2 is similar.

As the generalized solution of (1.3) we shall consider the pair . , v fulfilling the integral identities

$$(\psi, \varphi)_H = -\lambda\alpha \iint uv \frac{\partial}{\partial x} r^2\varphi \, dx\, dr - \iint \frac{1}{r^3} \Lambda\psi \frac{\partial (r^2\psi, r^2\varphi)}{\partial (x, r)} dx\, dr - \iint v^2 \frac{\partial}{\partial x} r^2\varphi \, dx\, dr$$

$$(v, \vartheta)_M = \iint r^3\psi \frac{\partial\vartheta}{\partial x} dx\, dr + \iint r^2 v \frac{\partial (r^2\psi, \vartheta)}{\partial (x, r)} dx\, dr, \qquad \varphi \in H,\ \vartheta \in M \qquad (2.6)$$

L e m m a 2.3 . Solution (2.3), (2.4) is the classical solution of problem (1.3).

As the classical solution of problem (1.3) we understand the pair of functions ψ, v which have continuous second (respectively first) derivatives in the closed space Ω , and also fourth (respectively second) continuous derivatives in Ω , and satisfying boundary conditions and Equations (1.3).

Proof of Lemma 2.3 is given in [4 and 5].

3. Each of the right-hand terms of system (2.3),(2.4) represents a linear limited functional with respect to $\varphi \in H\,(\vartheta \in M)$ when $\psi \in H,\ v \in M$. Consequently, in accordance with the theorem of the generalized form of the functional, we have in the Hilbertian space such $K_1uv,\ K_1v^2,\ K_2\psi,\ T\psi,\ Uv$, so that

$$(\psi, \varphi)_H = \lambda\alpha\,(K_1uv, \varphi)_H + (K_1v^2, \varphi)_H + (K_2\psi, \varphi)_H,\ \ (v, \vartheta)_M = (T\psi, \vartheta)_M + (Uv, \vartheta)_M$$

Therefore, system (1.3) is equivalent to the system of operator equations

$$\psi = \lambda\alpha K_1uv + K_1v^2 + K_2\psi \qquad (3.1)$$

$$v = T\psi + Uv \qquad (3.2)$$

L e m m a 3.1 . The operators at the right-hand sides of Equations (3.1) and (3.2) are fully continuous.

We shall prove this for the K_2 operator. By definition

$$(K_2\psi, \varphi)_H = -\iint \frac{1}{r^3} \Lambda\psi \frac{\partial}{\partial x} r^2\psi \frac{\partial}{\partial r} r^2\varphi \, dx\, dr + \iint \frac{1}{r^3} \Lambda\psi \frac{\partial}{\partial r} r^2\psi \frac{\partial}{\partial x} r^2\varphi \, dx\, dr \quad (3.3)$$

We evaluate the first term of (3.3). In accordance with Helder's inequality we have

$$\left| \iint \frac{1}{r^3} \Lambda\psi \frac{\partial}{\partial x} r^2\psi \frac{\partial}{\partial r} r^2\varphi \, dx\, dr \right| \leqslant \|\psi\|_H \left\| \frac{\partial\psi}{\partial x} \right\|_{L_4} \|\varphi\|_{W_4^1}$$

The second term is evaluated in the same manner.

Substituting $K_2\psi$ for φ and inserting W_2^2 into W_4^1 , we obtain

$$\| K_2\psi \|_H \leqslant C \|\psi\|_H \|\psi\|_{W_4^1} \qquad (3.4)$$

From this follows the boundedness. The continuity is proved in a similar manner, while from the complete continuity of the insertion of W_2^2 into W_4^1 follows the complete continuity of K_2.

We shall consider the homogeneous equation corresponding to (3.2)

$$v - Uv = 0 \qquad (3.5)$$

It has only a trivial solution $v = 0$.

This is proved by scalar multiplication of (3.4) by $r^2 v$ and integration by part

$$(v, r^2v)_M + \iint r^2 v \frac{\partial (r^2\psi, r^2 v)}{\partial (x, r)} dx\, dr = (v, r^2v)_M = \iint r \left[\left(\frac{\partial r^2 v}{\partial r} \right)^2 + \left(\frac{\partial r^2 v}{\partial x} \right)^2 \right] dx\, dr = 0$$

By virtue of Lemma 2.2 we have $v = 0$.

It follows from Fredholm's theorem that (3.2) can be solved for any $\psi \in H$, and consequently (3.2) defines operator

$$v = A\psi \qquad (3.6)$$

We shall prove that A is a fully continuous operator, acting from H in M . For this we carry out scalar multiplication of (3.5) by $r^2 v$. As previously we make use of Lemma 2.2, and obtain

$$\|v\|^2_M \leqslant C \iint r^3\psi \frac{\partial}{\partial x} r^2 v\, dx\, dr \leqslant C_1 \|\psi\|_{L_2} \|v\|_M, \quad \text{or} \quad \|v\|_M \leqslant C_1 \|\psi\|_{L_2}, \quad v = A\psi \quad (3.7)$$

In view of the complete continuity of insertion of W_2^2 in L_2 [6 and 7] it follows from here that operator A is completely continuous.

L e m m a 3.2 . We have the evaluation as follows $\|A\psi - T\psi\|_M \leqslant C\|\psi\|^2_H$.

By multiplying the two parts by Φ we have from (3.2)

$$(v - T\psi, \Phi)_M = \iint r^2 v \frac{\partial (r^2\psi, \Phi)}{\partial (x, r)}\, dx\, dr \leqslant C \|v\|_M \|\psi\|_H \|\Phi\|_M$$

assuming $v = A\psi$, $\Phi = A\psi - T\psi$ and taking into account (3.7) we obtain the proof of this Lemma.

4. By substituting into (3.1) the expression (3.6) for v, we reduce the solution of problem (1.3) to the solution of the operator equation

$$\psi = \lambda\alpha K_1 u A\psi + K_1 (A\psi)^2 + K_2\psi \equiv K(\psi, \lambda) \quad (4.1)$$

Similarly, the linear system (1.4) becomes

$$\psi = \lambda\alpha K_1 u T\psi \quad (4.2)$$

The right-hand side of (4.2) is the Freché differential of the right-hand side of (4.1). This statement follows immediately from the linearity of the operator K_1, Lemma 3.2 and evaluations (3.4) and (3.7).

We shall now use the bifurcation theory of nonlinear operator equations [8] for finding solutions of (1.3) different from those of (1.1).

The real number λ_1 is called the bifurcation point of operator K, if for any $\epsilon, \delta > 0$ the operator K characteristic number λ is such that $|\lambda - \lambda_1| < \epsilon$, and that (4.1) has at least one eigenfunction $\|\psi\|_H < \delta$.

It follows from Krasnosel'skii's investigations [8] that the bifurcation points of operator K can only be the characteristic numbers of its Freché differential (1.4).

If λ_1 is the prime characteristic number of problem (1.4), then it is the bifurcation point of operator K, and there is a continuous branch of operator K eigenvectors corresponding to this bifurcation point.

5. In the physical sense, parameter $\lambda = 4R^2$ is real and positive, therefore the bifurcation points of system (1.3) can only be the points of system (1.4) spectrum along the real axis. We shall limit our analysis to a unidirectional rotation of cylinders $(u > 0)$. There are two cases, namely $a^2\omega < 1$, and $a^2\omega > 1$.

Multiplying Equations (1.4) by ψ and $\lambda\alpha u r^{-1}$ respectively, integrating with respect to r and x, and taking into consideration boundary conditions and periodicity, it is not difficult to arrive at

$$\lambda = \left\{\int_1^a \int_0^{2\pi k^{-1}} \frac{1}{r^3} (\Lambda\psi)^2\, dx\, dr\right\}\left\{-\int_1^a \int_0^{2\pi k^{-1}} \alpha^2 r^{-1}\left(1 + \frac{\beta}{r^2}\right)\left[\left(\frac{\partial v}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial r}\right)^2\right] dx\, dr\right\}^{-1}$$

It follows from this that for $1 + \beta/r^2 > 0$ (or $a^2\omega > 1$) the eigen numbers are negative. In this case there are no bifurcation points in the system (1.3), and it would appear that (1.1) is the only stationary solution of Equations (1.3) of the periodic function type. This corresponds to the results obtained by Synge [9] as regards the steady flow stability (1.1) for $a^2\omega > 1$.

We shall demonstrate that for $a^2\omega < 1$ (or $\alpha u > 0$) all eigen numbers of (1.3) are bifurcation points of nonlinear equations of steady state motion (1.3).

We shall apply to (1.4) the Fourier transformation with finite limits [10]. This is formally equivalent to the substitution into (1.4) of Fourier sine and cosine series expansions of ψ and v, respectively

$$\psi \sim \Sigma\psi_n(r)\sin nkx,\quad v \sim \Sigma\, v_n(r)\cos nkx$$

For the determination of the Fourier coefficients we have a system of equations consisting of pairs as follows:

$$\Lambda_1 \frac{1}{r^3}\Lambda_1\psi_n = \lambda nk\alpha r^3 u v_n,\quad -\Lambda_1 v_n = knr^3\psi_n,\quad v_n = \psi_n = \psi_n{}' = 0 \quad \text{for} \quad r = 1$$

$$\left(\Lambda_1 = \frac{d}{dr} r^3 \frac{d}{dr} + r^3k^2n^2\right) \qquad (n = 1, 2, \ldots) \tag{5.1}$$

Equation $\Lambda_1 z = 0$ has a positive solution $z = I_1(kr)/r$ where I_1 is the Bessel function of a purely imaginary argument [11]. By substituting variables

$$\psi_n = z\varphi,\quad v_n = t_0{}^2kz\vartheta,\quad t = t_0{}^{-1}\int_1^r \frac{dr}{r^3z^2},\quad r^6z^4 = J^{-1}(t),\quad t_0 = \int_1^a \frac{dr}{r^3z^2}$$

we obtain system (5.1) in the form

$$\frac{d^2}{dt^2} J\frac{d^2\varphi}{dt^2} = \lambda^2(Jr)^{-1}\alpha u\vartheta,\quad -\frac{d^2\varphi}{dt^2} = J^{-1}\varphi,\quad v = \varphi = \varphi' = 0 \quad \text{for} \quad t = 0, 1 \tag{5.2}$$

Operators at the left-hand side of (5-2) have been thoroughly studied in connection with stability problems of the string and rod rigidly fixed at their ends [12]. We shall reduce (5.2) with the aid of Green's functions to the integral equation

$$\varphi(t) = \lambda\int_0^1 S(t,\sigma)\varphi(\sigma)\,d\sigma \tag{5.3}$$

with a positive kernel for $a^2\omega < 1$ (or $\alpha u > 0$)

$$S(t,\sigma) = \int_0^1 (rJ)^{-1}\alpha u G_\varphi(t,s)\,G_\vartheta(s,\sigma)\,J^{-1}(\sigma)\,ds \qquad (S(t,\sigma) > 0)$$

Green's functions G_φ and G_ϑ are continuous, symmetric and oscillatory functions [12].

It is easy to prove that kernel $S(t, \sigma)$ is oscillatory. For this it is necessary to check whether the following properties are fulfilled (notations are those used in [12]

$$(1) \qquad S\begin{pmatrix} t_1 \ldots t_n \\ \sigma_1 \ldots \sigma_n \end{pmatrix} \geqslant 0 \qquad \begin{pmatrix} 0 < t_1 < \ldots < t_n < 1, & n = 1, 2\ldots \\ 0 < \sigma_1 < \ldots < \sigma_n < 1, & \end{pmatrix}$$

$$(2) \qquad S\begin{pmatrix} t_1 \ldots t_n \\ t_1 \ldots t_n \end{pmatrix} > 0 \qquad (0 < t_1 < \ldots < t_n < 1,\quad n = 1, 2\ldots)$$

The first property easily follows from the Bigné-Cauchy formula for matrix construction, after substitution of integral by summation. We shall prove the second property. We substitute integral by summation dividing the interval (0, 1) into m equal parts and we obtain

$$S\begin{pmatrix} t_1 \ldots t_n \\ t_1 \ldots t_n \end{pmatrix} = \lim_{m\to\infty} \frac{1}{m^n} \det\{\| g(s_j)\, G_\varphi(t_i, s_j) \| \| J^{-1}(t_i)\, G_\vartheta(s_j, t_i) \|\}$$

$$(i = 1, \ldots, n;\ j = 1, \ldots, m;\ m > n;\ g(t) = \alpha r^{-1}(t)\, J^{-1}(t)\, u\,(t))$$

The Bigné-Cauchy formula states that the determinant of product of matrices is equal to the sum of products of multiplication of corresponding minors of the multiplied matrices

$$S\begin{pmatrix} t_1 \ldots t_n \\ t_1 \ldots t_n \end{pmatrix} = \lim_{m\to\infty} \frac{1}{m^n} \Sigma g(s_{i_1}) \ldots g(s_{i_n})\, G_\varphi\begin{pmatrix} t_1 \ldots t_n \\ s_{i_1} \ldots s_{i_n} \end{pmatrix} \times \tag{5.4}$$

$$\times\, G_\vartheta\begin{pmatrix} s_{i_1} \ldots s_{i_n} \\ t_1 \ldots t_n \end{pmatrix} J^{-1}(t_1) \ldots J^{-1}(t_n)$$

For any arbitrary oscillatory function determinants

$$G\begin{pmatrix} t_1 \ldots t_n \\ s_1 \ldots s_n \end{pmatrix} > 0 \quad \text{for } 0 < t_1, s_1 < \ldots < t_n, s_n < 1 \tag{5.5}$$

The remaining determinants of the right-hand side of (5.4) are nonnegative.

We encircle each t_i by an interval of length $l < \min(t_{i+1} - t_i)$. In each of these there will be $\geq ml - 2$ points s_i which satisfy condition (5.5). The number of determinants in the right-hand side of (5.4) which are not zeros will then be $\geq (ml - 2)^n$. Thus,

$$S\begin{pmatrix} t_1 \ldots t_n \\ t_1 \ldots t_n \end{pmatrix} \geqslant \inf (gJ^{-1})^n \lim_{m\to\infty} \frac{1}{m^n} (ml-2)^n \inf G_\varphi \begin{pmatrix} t_1 \ldots t_n \\ s_1 \ldots s_n \end{pmatrix} G_\vartheta \begin{pmatrix} t_1 \ldots t_n \\ s_1 \ldots s_n \end{pmatrix} =$$

$$= l^n \inf (gJ^{-1})^n \inf G_\varphi \begin{pmatrix} t_1 \ldots t_n \\ s_1 \ldots s_n \end{pmatrix} G_\vartheta \begin{pmatrix} s_1 \ldots s_n \\ t_1 \ldots t_n \end{pmatrix}$$

where the exact lower limit is taken with respect to all s_i pertaining to the aforementioned intervals. (Because of the continuity of G_φ and G_ϑ this is reached with some of s_i which satisfy condition (5.5)).

Integral equation (5.3), as an equation having an oscillatory kernel, has positive and prime eigen numbers

$$0 < \lambda_1(nk) < \ldots < \lambda_s(nk) \to \infty$$

Each eigen number $\lambda = \lambda_1(nk)$ of some of the pairs of Equations (5.1) will be an eigen number of (1.4) with its eigenfunction $\psi = \psi_{ns}(r) \sin nkx$; $v = v_{ns}(r) \cos nkx$. It will be a prime eigen number of (1.4) in $H(k)$, $N(k)$ and, therefore a point of bifurcation of (1.3) in $H(k)$, $N(k)$, provided there is no $\lambda_p(mk) = \lambda_s(nk)$, otherwise it will be a multiple eigen number (*) of system (1.4) in $H(k)$, $N(k)$ (without loss of generality, it can be assumed that $m > n$). There will be no associated eigenvectors in system (1.4), as the assumption of their existence leads to the conclusion that such vectors also exist in system (5.1), which is not possible. There may exist a certain number of such m, but their number is finite, because due to the complete continuity of operator T in (4.2) each eigen number of (1.4) is of finite multiplicity. This eigen number will, however, be a prime number for (1.4) and, consequently, a bifurcation point for (1.3) in $H(m^*k)$, $N(m^*k)$, where m^* is the largest of m.

We shall note certain properties [12] of the system (1.4) eigenfunctions. For this we introduce the necessary definitions. An individual null, or a continuous null interval $\psi(r)$ is called the null area of function $\psi(r)$. This null area is called the nodal area, if function $\psi(r)$ changes its sign when passing through it.

1. The first of the eigenfunctions $\psi_{n1}(r)$, $v_{n1}(r)$ of system (5.1) do not contain zeros within the interval $(1, a)$.

2. For any $s = 1, 2, \ldots$ the eigenfunctions have exactly $s - 1$ nodal areas, and there are no other zeros in the interval $(1, a)$.

3. For any integer values of l and $m (m \leq l)$ the linear combinations

$$\sum_{s=l}^{m} c_s \psi_{ns}, \quad \sum_{s=l}^{m} b_s v_{ns} \quad \left(\sum_{s=l}^{m} c_s^2, \ \sum_{s=l}^{m} b_s^2 > 0\right)$$

have not less than s nodal areas and not more than m null areas in the interval $(1, a)$.

4. The nodal points of two adjacent fundamental functions $\psi_{ns}(r)$ and $\psi_{ns+1}(r)$, (v_{ns} and v_{ns+1}) alternate.

---

*) This apparently takes place when certain relation between the period and the distance between the cylinder walls reaches rational values.

Properties of the eigenfunctions v_{ns} follow from the integral equation for v_s similarly to (5.3).

Due to the smallness of the norm of solutions of the nonlinear system (1.3), solutions of the latter in the neighborhood of bifurcation points are approximated by solutions of the linear system (1.4), namely $\psi = \psi_{ns}(r)\sin nkx$, $v = v_{ns}(r)\cos nkx$. The properties enumerated above indicate a complex fluid flow pattern, as was observed by Taylor in his experiments [2 and 3]. The streamlines of the perturbed flow follow toroidal surfaces contained between the cylinder walls. The number of such surfaces increases with the increase of R.

BIBLIOGRAPHY

1. Kochin, N.E., Kibel', I.A. and Roze, N.V., Teoreticheskaia gidromekhanika (Theoretical Hydromechnics). Pt.II, Fizmatgiz, 1963.

2. Taylor, G.I., Experiments with rotating fluids. Proc.Int.Congr.appl. Mech., Delft, pp.86-96, 1924.

3. Taylor, G.I., Stability of viscous liquid contained between two rotating cylinders. Proc.Roy.Soc.A., Vol. 223, p.289, 1923.

4. Ladyzhenskaia, O.A., Issledovanie uravneniia Nav'e-Stoksa v sluchae statsionarnogo dvizheniia neszhimaemoi zhidkosti (Investigation of the Navier-Stokes equation in the case of steady state motion of an incompressible fluid). Usp.mat.Nauk, Vol.14, № 3 (87), 1959.

5. Vorovich, I.I. and Iudovich, V.I., Statsionarnoe techenie viazkoi neszhimaemoi zhidkosti (Steady state flow of a viscous incompressible fluid). Mat.Sb., Vol.53 (25), № 4, 1961.

6. Sobolev, S.L., Nekotorye primeneniia funktsional'nogo analiza v matematicheskoi fizike (Some Applications of Functional Analysis in Mathematical Physics). Izd.Leningrad.Univ., 1950.

7. Smirnov, V.I., Kurs vysshei matematiki (Course of Higher Mathematics). Vol.V, Fizmatgiz, 1959.

8. Krasnosel'skii, M.A., Topologicheskie metody v teorii nelineinykh integral'nykh uravnenii (Topological Methods in the Theory of Nonlinear Integral Equations). Gostekhizdat, 1956.

9. Lin',C.C., Teoria gidrodinamicheskoi ustoichivosti (Theory of Hydrodynamic Stability). Izd.inostr.Lit., 1956.

10. Sneddon, I.N., Preobrazovanie Fur'e (The Fourier Transformation). Izd. inostr.Lit., 1955.

11. Jahnke, E., Emde, F. and Lesch, F., Spetsial'nye funktsii (Special Functions). Izd.Nauka, 1964.

12. Gantmacher, F.R. and Krein, M.G., Ostsilliatsionnye matritsy i malye kolebaniia mekhanicheskikh sistem (Oscillatory Matrices and Small Oscillations of Mechanical Systems). Gostekhizdat, 1950.

Translated by J.J.D.